3.4(2) and 3.5 Operations and Inverses

Matrix algebra: $\underline{A}, \underline{B}, \underline{C}$ all $(n \times n)$

1) $\underline{A} + \underline{B} = \underline{B} + \underline{A}$, 2) $\underline{A}(\underline{B} + \underline{C}) = \underline{AB} + \underline{AC}$

3) $(\underline{A} + \underline{B})\underline{C} = \underline{AC} + \underline{BC}$ 4) $\underline{A}(\underline{BC}) = (\underline{AB})\underline{C}$

* 5) k scalar, $\underline{x}$ vector. $\underline{A}(k\underline{x}) = k(\underline{A}\underline{x}) = (k\underline{A})\underline{x}$

("scalars factor through") $\underline{A}(k\underline{B}) = k(\underline{A}\underline{B})$ ...

$$\underset{\underline{A}}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \underset{\underline{I}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \underline{A}$$

$$\underline{A}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\underline{A}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} : \begin{array}{l} (\text{row } 1) \to \begin{bmatrix} 1 & 2 \end{bmatrix} \\ (\text{row } 2) \to \begin{bmatrix} 3 & 4 \end{bmatrix} \end{array}$$

so $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\underline{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \underline{A}$

Identity matrices "1's on main diagonal"

$\underline{I_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\underline{I_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\underline{I_4} = \cdots$

Often just write "$\underline{I}$" when size is clear

For all matrices $\underline{A}$, $\boxed{\underline{AI} = \underline{IA} = \underline{A}}$ $(\underline{I}\underline{x} = \underline{x})$

(assuming products are well defined)

(so $\underline{I}$ is like "1" for matrices)

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}}_{\underline{B}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\underline{AB} = \underline{I}}$$

$$\underbrace{\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}}_{\underline{B}} \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{\underline{A}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\underline{BA} = \underline{I}}$$

Def: Since $\underline{AB} = \underline{BA} = \underline{I}$, we say $\underline{A}$ "invertible", (or "nonsingular")

$\underline{B} = \underline{A}^{-1}$ ("$\underline{A}$ inverse")

(similarly for $\underline{B}$, $\underline{B}^{-1}$)

- Only square (2×2, 3×3, $\cdots$) matrices can have inverses
- Not every $\underline{A}$ has $\underline{A}^{-1}$ defined.
- for example $\emptyset$ – matrices $\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\underline{\emptyset_2}}$ $\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\underline{\emptyset_3}}$, $\cdots$

  $\underline{\emptyset}^{-1}$ undefined.
- If $\underline{A}^{-1}$ undefined, $\underline{A}$ is "singular" or "non-invertible"

Special 2×2 formula $\underline{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- $\det(\underline{A}) = |\underline{A}| \overset{\text{DEF}}{=} ad - bc$. (the "determinant" of $\underline{A}$)

★ $\boxed{\underline{A} \text{ invertible} \iff |\underline{A}| \neq 0}$

If $|\underline{A}| \neq 0$, then $\underline{A}^{-1} = \dfrac{1}{|\underline{A}|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\underline{\mathcal{E}x}$ $\quad \underline{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $|\underline{A}| = 1\cdot 4 - 2\cdot 3 = -2 \ (\neq 0)$

$$\underline{A}^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Solving systems with "inverse method"

$$\begin{cases} x_1 + 3x_2 = 9 \\ 2x_1 + x_2 = 8 \end{cases}$$

same as $\quad x_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}}_{\underline{A}}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\underline{x}} \underset{=}{=} \underbrace{\begin{bmatrix} 9 \\ 8 \end{bmatrix}}_{\underline{b}}$$

Idea: • IF $\underline{A}^{-1}$ defined, then

$$\underline{Ax} = \underline{b} \to \underline{A^{-1}Ax} = \underline{A^{-1}b}$$
$$\to \underline{Ix} = \underline{A^{-1}b} \to \boxed{\underline{x} = \underline{A}^{-1}\underline{b}}$$

• $|\underline{A}| = 1\cdot 1 - 3\cdot 2 = -5 \neq 0;$

$$\underline{A}^{-1} = \frac{1}{-5}\begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1/5 & 3/5 \\ 2/5 & 1/5 \end{bmatrix}$$

$$\underline{A}^{-1}\,\underline{Ax} = \underline{A^{-1}b} \to \underline{x} = \underline{A^{-1}b}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 8 \end{bmatrix} = \frac{-1}{5}\left(\begin{bmatrix} 9 \\ -18 \end{bmatrix} + \begin{bmatrix} -24 \\ 8 \end{bmatrix}\right)$$

$$= \frac{-1}{5}\begin{bmatrix} -15 \\ -10 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

How to compute $\underline{A}^{-1}$ if possible. (Ex: 3x3)

Try $\underline{A} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 5 \end{bmatrix}$

$$\left[\underline{A} \,\vdots\, \underline{I_3}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 2 & 4 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{pmatrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_2 \end{pmatrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right]$$

$$(R_3 \to R_3 - 4R_2) \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 5 & 10 & -4 & 1 \end{array}\right]$$

$$(R_3 \to \tfrac{1}{5} R_3) \left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \;\middle|\; \underbrace{\begin{matrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & -\frac{4}{5} & \frac{1}{5} \end{matrix}}_{\underline{A}^{-1}}\right]$$

- Start with $\underline{A}$ $n \times n$,
- Make matrix $\left[\underline{A} \,\vdots\, \underline{I_n}\right]$
- Do row operations to reduce $\underline{A}$ to RREF

<u>Case 1:</u> $(\text{RREF } \underline{A}) = \underline{I_n}$

then result is $\left[\underline{I_n} \,\vdots\, \underline{A}^{-1}\right]$

<u>Case 2:</u> $(\text{RREF } \underline{A}) \neq I_n$

then $\underline{A}$ is not invertible.

Ex $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$

$\underbrace{}_{A}$

$\begin{pmatrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix} \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right]$

no pivot in R_2, so no hope to reduce to I_3.

Thus, A not invertible

Why does $[A \mid I] \to [I \mid A^{-1}]$ work?

Idea: • Pick B so that $BA = (\text{RREF } A)$
("what are row operations"?)

• Row ops $\longleftrightarrow$ multiplying by B.

• $B[A \mid I] = [BA \mid BI]$
$= [I \mid B]$
$= [I \mid A^{-1}]$

More on inverses

• $(A^{-1})^{-1} = A$

• inverses unique.

• $\boxed{(AB)^{-1} = B^{-1}A^{-1}}$ NOT $A^{-1}B^{-1}$

Matrix powers: $A^k = \underbrace{A A \cdots A}_{k \text{ times}}$